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Shallow water approximations for water waves

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1 Introduction

In this communication we are concerned with the initial value problem for two types of water waves and their shallow water approximations. The first type of the water wave is the standard one, that is, the fluid is bounded from above by a free surface and from below by a rigid boundary, and is subject to a uniform gravity in the vertical direction as an external force. This type of problem will be referred as Problem I in the following. The second type of the water wave corresponds to the ocean around the earth, that is, we take an effect of the curvature into account on the surface of the earth. Therefore, the free surface and the bottom are nearly spheres and the fluid is subject to the gravitation due to the earth. This type of problem will be referred as Problem II in the following.

The water wave is a model for an irrotational flow of an incompressible ideal fluid with a free surface under the gravitational field. The analysis of this problem is very hard because of the nonlinearity of the equations together with the presence of an unknown free surface. In order to understand various phenomena of water waves, one has approximated the equations by simple ones and analyzed the approximated equations. The simplest approximation is the linear one around the trivial flow by assuming that the amplitude of the free surface and the motion of the fluid are infinitesimal. However, this approximation could not explain the existence of solitary waves nor the breaking of water waves. In order to explain such phenomena we have to include nonlinear effects of the waves in the approximation. The shallow water equations are one of such approximations and derived from the water wave by assuming that the water depth is sufficiently small compared to the wave length. The aim of this communication is to report a recent result of mathematically rigorous justification of the shallow water approximation for water waves, especially a justification in Sobolev spaces.

Mathematically, the problem is formulated as a free boundary problem for incompressible Euler equation with the irrotational condition. By rewriting the equations for water waves in a non-dimensional form, we have a non-dimensional parameters δ the ratio of the water depth h to the wave length λ in Problem I and to the mean radius R of the earth in Problem II, respectively, in the equations. The shallow water equations are derived from the water wave in the limit $\delta \to +0$. In the case of a flat bottom in Problem I, they are of the same form as the compressible Euler equation for a barotropic gas and the solution generally has a singularity in finite time even if the initial data are sufficiently smooth. Therefore, this approximation is used to explain the breaking of water waves. The derivation of the shallow water equations goes back to G.B. Airy [1]. Then, K.O. Friedrichs [3] derived systematically the equations from the water wave problem by using an expansion of the solution with respect to δ^2, which is called the Friedrichs expansion. A mathematically rigorous justification of the shallow water approximation for 2-dimensional water waves was given by L.V. Ovsjannikov [11, 12] under the periodic boundary condition with respect to the horizontal spatial variable, and then by T. Kano and T. Nishida [6]. A mathematical justification of the Friedrichs expansion was investigated by T. Kano and T. Nishida [7] and the justification in the 3-dimensional case by T. Kano [5]. In order to guarantee the existence of solutions for water waves, they used an abstract Cauchy-Kowalevski theorem in a scale of Banach spaces so that analyticity of the initial data was required. It is natural to ask if the approximation is valid in Sobolev spaces. However, this question was not resolved for long time.

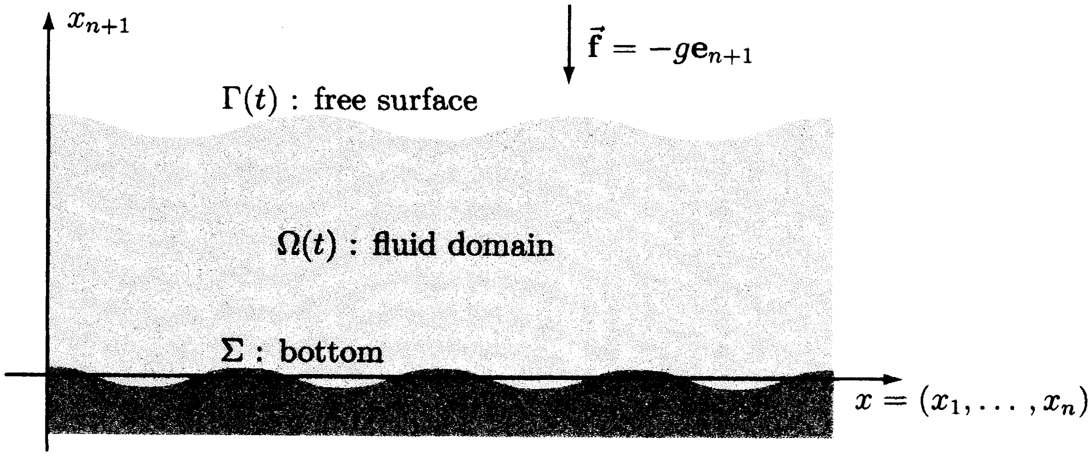
In connection with the well-posedness of the initial value problem for water waves, the solvability in Sobolev spaces was given by several authors. In his pioneering work [10], V.I. Nalimov investigated the initial value problem in the case where the motion of the fluid is 2-dimensional and the fluid has infinite depth. He showed that if the initial data are sufficiently small in a Sobolev space, that is, if the initial surface is almost flat and the initial movement of the fluid is sufficiently small, then there exists a unique solution of the problem locally in time in a Sobolev space. H. Yosihara [16] extended this result to the case of presence of an almost flat bottom. S. Wu [14] studied the problem in exactly the same situation as Nalimov's and gave the existence theorem locally in time without assuming the initial data to be small. It is known that the well-posedness of the problem may be broken unless a generalized Rayleight-Taylor sign condition $-\partial p/\partial N \geq c_0 > 0$ on the free surface is satisfied, where p is the pressure and N is the unit outward normal to the free surface. She showed that this condition always holds for any smooth nonself-intersecting surface. S. Wu [15] also succeeded in giving an existence theory in Sobolev spaces for 3-dimensional water waves of infinite depth. Note that all of the three authors mentioned above used the Lagrangian coordinates. D. Lannes [8] studied the initial value problem for water waves of finite depth in arbitrary space dimensions. One of interesting features of his paper is that he did not use the Lagrangian coordinates but the Euler coordinates although the surface tension on the free surface was neglected. Another interesting feature is that he obtained a good expression of the Fréchet derivative of the Dirichlet-to-Neumann map for Laplace's equation with respect to a function which represents the surface elevation. As a result, he derived nice linearized equations and succeeded in giving an existence theory in Sobolev spaces.

The existence theories in Sobolev spaces were based on the energy method. In calculation of the time evolution of an energy function, we need to estimate commutators of the Dirichlet-to-Neumann map and differential operators. S. Wu [15] obtained precise commutator estimates by using the theory of singular integral operators and Clifford analysis, whereas D. Lannes [8] used the theory of pseudo-differential operators and obtained commutator estimates by imposing much differentiability on the coefficients. This is one of the reasons why a Nash-Moser implicit function theorem was used to obtain the solution of the nonlinear equations in [8]. A relation between the generalized Rayleight-Taylor sign condition and the bottom topography was also analyzed in [8]. Under a shallow water regime $\delta \ll 1$, such techniques in [15, 8] in estimating commutators do not give nice uniform estimates with respect to small δ. In this communication, to obtain the uniform estimates, we only use the standard technique in estimating the solution of a boundary value problem for elliptic differential equations, so that the proof may become much simpler and more elementary than the previous ones. We adopt the formulation of the problem used in [8]. However, thanks to a precise energy estimate for linearized equations and a reduction of the full nonlinear equations to a system of quasilinear equations, we do not use the Nash-Moser implicit function theorem to obtain the solution of the nonlinear equations.

Recently, Y.A. Li [9] considered a shallow water approximation for 2-dimensional water waves over a flat bottom and gave a mathematical justification of the approximation by the Green-Naghdi equations in Sobolev spaces. His method depends deeply on the use of a conformal map, so that it is restricted to the 2-dimensional case. Then, B. Alvarez-Samaniego and D. Lannes [2] and the author [4] gave a justification of the shallow water approximation for 3-dimensional water waves in Sobolev spaces. In [2] they gave also justifications of several asymptotic models for 3-dimensional water waves including the Kadomtsev-Petviashvili (KP) equation. However, they still used the Nash-Moser implicit function theorem, whereas we do not use the theorem in this communication. All of the results mentioned above were concerned with Problem I and it seems to the author that there is no mathematically rigorous result on Problem II.

2 Formulation of Problem I

The first type of the water wave is the standard one and the shape of the fluid region is shown in the following illustration.



Let $x = (x_1, x_2, \dots, x_n)$ be the horizontal spatial variables and x_{n+1} the vertical spatial variable. We denote by $X = (x, x_{n+1}) = (x_1, \dots, x_n, x_{n+1})$ the whole spatial variables. We will consider a water wave in $(n+1)$-dimensional space and assume that the domain $\Omega(t)$ occupied by the fluid at time $t \geq 0$, the free surface $\Gamma(t)$, and the bottom Σ are of the forms

$$\begin{aligned} \Omega(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, b(x) < x_{n+1} < h + \eta(x,t)\}, \\ \Gamma(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = h + \eta(x,t)\}, \\ \Sigma &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = b(x)\}, \end{aligned}$$

where h is the mean depth of the fluid. The functions b and η represent the bottom topography and the surface elevation, respectively. In this problem b is a given function, while η is the unknown. In fact, our main interest is the behavior of the free surface, so that we have to study the behavior of this function η.

We assume that the fluid is incompressible and inviscid, and that the flow is irrotational. Then, the fluid motion is described by the velocity potential $\Phi = \Phi(X,t)$ satisfying the equation

$$\Delta_X \Phi = 0 \quad \text{in} \quad \Omega(t), \quad t > 0, \tag{2.1}$$

where Δ_X is the Laplacian with respect to X, that is, $\Delta_X = \Delta + \partial_{n+1}^2$ and $\Delta = \partial_1^2 + \cdots + \partial_n^2$. The boundary conditions on the free surface are given by

$$\begin{cases} \eta_t + \nabla\Phi \cdot \nabla\eta - \partial_{n+1}\Phi = 0, \\ \Phi_t + \frac{1}{2}|\nabla_X \Phi|^2 + g\eta = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0, \end{cases} \tag{2.2}$$

where $\nabla = (\partial_1, \dots, \partial_n)^T$ and $\nabla_X = (\partial_1, \dots, \partial_n, \partial_{n+1})^T$ are the gradients with respect to $x = (x_1, \dots, x_n)$ and to $X = (x, x_{n+1})$, respectively, and g is the gravitational constant. The first equation is the kinematical condition and the second one is what is known as Bernoulli's law restricted on the free surface. The boundary condition on the bottom is given by

$$N \cdot \nabla_X \Phi = 0 \qquad \text{on} \quad \Sigma, \quad t > 0, \tag{2.3}$$

where N is the normal vector to the bottom Σ. Finally, we impose the initial conditions

$$\eta(x, 0) = \eta_0(x), \quad \Phi(X, 0) = \Phi_0(X). \tag{2.4}$$

It should be assumed that the initial data satisfy the compatibility conditions, that is, $\Delta_X \Phi_0 = 0$ in $\Omega(0)$ and $N \cdot \nabla_X \Phi_0 = 0$ on Σ.

Remark 2.1. In a derivation of the second equation in (2.2) we first integrate the conservation of momentum, that is, the Euler equation $0 = \rho\big(v_t + (v \cdot \nabla_X)v\big) + \nabla_X p + \rho g \mathbf{e}_{n+1} = \rho \nabla_X \big(\Phi_t + \frac{1}{2}|\nabla_X \Phi|^2 + \frac{1}{\rho}(p - p_0) + g(x_{n+1} - h)\big)$ and obtain

$$\Phi_t + \tfrac{1}{2}|\nabla_X \Phi|^2 + \tfrac{1}{\rho}(p - p_0) + g(x_{n+1} - h) = f(t) \qquad \text{in} \quad \Omega(t), \quad t > 0,$$

where $v = \nabla_X \Phi$ is a velocity, ρ is a constant density, p_0 is a constant atmospheric pressure, $\mathbf{e}_{n+1}$ is the unit vector in the vertical direction, and $f(t)$ is an arbitrary function of time t. This equation expresses what is called Bernoulli's law. Replacing Φ by $\Phi + \int f(t)dt$, restricting the above equation on the free surface $\Gamma(t)$, and using the dynamical boundary condition $p = p_0$ on $\Gamma(t)$, we get the second equation in (2.2).

We proceed to rewrite the equations (2.1)–(2.4) in an appropriate non-dimensional form. Let λ be the typical wave length and h the mean depth. We introduce a non-dimensional parameter

$$\delta := \frac{h}{\lambda}$$

that represents the shallowness of the water, and rescale the independent and dependent variables by

$$x = \lambda \tilde{x}, \quad x_{n+1} = h \tilde{x}_{n+1}, \quad t = \frac{\lambda}{\sqrt{gh}} \tilde{t}, \quad \Phi = \lambda \sqrt{gh} \tilde{\Phi}, \quad \eta = h \tilde{\eta}, \quad b = h \tilde{b}.$$

Putting these into (2.1)–(2.4) and dropping the tilde sign in the notation we obtain

$$\delta^2 \Delta \Phi + \partial_{n+1}^2 \Phi = 0 \qquad \text{in} \quad \Omega(t), \quad t > 0, \tag{2.5}$$

$$\begin{cases} \delta^2\big(\eta_t + \nabla \Phi \cdot \nabla \eta\big) - \partial_{n+1} \Phi = 0, \\ \delta^2\big(\Phi_t + \frac{1}{2}|\nabla \Phi|^2 + \eta\big) + \frac{1}{2}(\partial_{n+1}\Phi)^2 = 0 \qquad \text{on} \quad \Gamma(t), \quad t > 0, \end{cases} \tag{2.6}$$

$$\partial_{n+1}\Phi - \delta^2 \nabla b \cdot \nabla \Phi = 0 \qquad \text{on} \quad \Sigma, \quad t > 0, \tag{2.7}$$

$$\eta(x,0) = \eta_0^\delta(x), \quad \Phi(X,0) = \Phi_0^\delta(X), \tag{2.8}$$

where

$$\begin{aligned}
\Omega(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, b(x) < x_{n+1} < 1 + \eta(x,t)\},\\
\Gamma(t) &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = 1 + \eta(x,t)\},\\
\Sigma &= \{X = (x, x_{n+1}) \in \mathbf{R}^{n+1}\,;\, x_{n+1} = b(x)\}.
\end{aligned}$$

Since we are interested in asymptotic behavior of the solution when $\delta \to +0$, we always assume $0 < \delta \leq 1$ in the following.

As in the usual way, we transform equivalently the initial value problem (2.5)–(2.8) to a problem on the free surface. To this end, we introduce new unknown function ϕ by

$$\phi(x,t) := \Phi(x, 1 + \eta(x,t), t), \tag{2.9}$$

which is the trace of the velocity potential on the free surface. Then, we see that

$$\begin{cases} \phi_t = \Phi_t|_{\Gamma(t)} + \partial_{n+1}\Phi|_{\Gamma(t)}\eta_t, \\ \nabla\phi = \nabla\Phi|_{\Gamma(t)} + \partial_{n+1}\Phi|_{\Gamma(t)}\nabla\eta. \end{cases} \tag{2.10}$$

It follows from (2.5), (2.7), and (2.9) that

$$\Lambda(\eta, b, \delta)\phi = (\delta^{-2}\partial_{n+1}\Phi - \nabla\eta \cdot \nabla\Phi)\big|_{\Gamma(t)}, \tag{2.11}$$

where $\Lambda = \Lambda(\eta, b, \delta)$ is a linear operator called the Dirichlet-to-Neumann map for Laplace's equation. More precisely, the Dirichlet-to-Neumann map is defined in the following way.

Definition 2.1. Under appropriate assumptions on η and b, for any function φ on the free surface in some class there exists a unique solution Φ of the boundary value problem

$$\begin{cases} \delta^2\Delta\Phi + \partial_{n+1}^2\Phi = 0 & \text{in} \quad b(x) < x_{n+1} < 1 + \eta(x), \\ \Phi = \varphi & \text{on} \quad x_{n+1} = 1 + \eta(x), \\ \partial_{n+1}\Phi - \delta^2\nabla b \cdot \nabla\Phi = 0 & \text{on} \quad x_n = b(x). \end{cases}$$

Using the solution Φ we define a linear operator $\Lambda = \Lambda(\eta, b, \delta)$ by

$$\Lambda(\eta, b, \delta)\varphi := (\delta^{-2}\partial_{n+1}\Phi - \nabla\eta \cdot \nabla\Phi)\big|_{\Gamma(t)}.$$

This operator Λ maps the Dirichlet data to the Neumann data on the free surface, so that it is called the Dirichlet-to-Neumann map. Hereafter, the solution Φ is denoted by $\varphi^\hbar$.

The second equation in (2.10) and (2.11) imply that

$$\begin{cases} \partial_{n+1}\Phi|_{\Gamma(t)} = \delta^2(1 + \delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta \cdot \nabla\phi), \\ \nabla\Phi|_{\Gamma(t)} = \nabla\phi - \delta^2(1 + \delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta \cdot \nabla\phi)\nabla\eta. \end{cases} \tag{2.12}$$

It follows from the first equation in (2.6) and (2.11) that $\eta_t - \Lambda\phi = 0$, so that by the first equation in (2.10) we get

$$\Phi_t|_{\Gamma(t)} = \phi_t - \delta^2(1+\delta^2|\nabla\eta|^2)^{-1}(\Lambda\phi + \nabla\eta\cdot\nabla\phi)\Lambda\phi.$$

Putting this and (2.12) into the second equation in (2.6) we obtain

$$\text{(2.13)}\qquad \begin{cases} \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}\delta^2(1+\delta^2|\nabla\eta|^2)^{-1}\big(\Lambda(\eta,b,\delta)\phi + \nabla\eta\cdot\nabla\phi\big)^2 = 0, \\ \eta_t - \Lambda(\eta,b,\delta)\phi = 0 \qquad \text{for} \quad t>0, \end{cases}$$

$$\text{(2.14)}\qquad \eta = \eta_0^\delta, \quad \phi = \phi_0^\delta \qquad \text{at} \quad t=0,$$

where $\phi_0^\delta = \Phi_0^\delta(\cdot, 1+\eta_0^\delta(\cdot))$. This is one of the initial value problems that we are going to investigate in this communication. The following theorem asserts the existence of the solution to the above initial value problem with uniform bounds of the solution on a time interval independent of small $\delta > 0$.

Theorem 2.1 ([4]). *Let $M_0, c_0 > 0$ and $s > n/2+1$. There exist a time $T>0$ and constants $C_0, \delta_0 > 0$ such that for any $\delta \in (0,\delta_0]$, $\nabla\phi_0^\delta \in H^{s+3}$, $\eta_0^\delta \in H^{s+3+1/2}$, and $b \in H^{s+4+1/2}$ satisfying*

$$\begin{cases} \|\nabla\phi_0^\delta\|_{s+3} + \|\eta_0^\delta\|_{s+3+1/2} + \|b\|_{s+4+1/2} \le M_0, \\ 1+\eta_0^\delta(x) - b(x) \ge c_0 \quad \textit{for} \quad x\in\mathbf{R}^n, \end{cases}$$

the initial value problem (2.13) and (2.14) has a unique solution $(\eta,\phi) = (\eta^\delta,\phi^\delta)$ on the time interval $[0,T]$ satisfying

$$\begin{cases} \|\eta^\delta(t)\|_{s+3} + \|\nabla\phi^\delta(t)\|_{s+2} + \|(\eta_t^\delta(t),\phi_t^\delta(t))\|_{s+2} \le C_0, \\ 1+\eta^\delta(x,t) - b(x) \ge c_0/2 \quad \textit{for} \quad x\in\mathbf{R}^n,\ 0\le t\le T,\ 0<\delta\le\delta_0. \end{cases}$$

Remark 2.2. We cannot expect that the velocity potential Φ and its trace ϕ on the free surface vanish at spatial infinity even if so does the velocity $v = \nabla_X\Phi$. Hence, it is natural to consider the initial value problem (2.13) and (2.14) in a class $\nabla\phi \in H^s$ (not a class $\phi\in H^s$). However, if we impose additional conditions $\phi_0^\delta \in L^2$ and $\|\phi_0^\delta\| \le M_0$, then we have $\phi^\delta \in C([0,T];H^{s+3})$ with a uniform estimate $\|\phi^\delta(t)\|_{s+3} \le C_0$.

3 Shallow water approximation for Problem I

We proceed to study formally asymptotic behavior of the solution $(\eta^\delta,\phi^\delta)$ to the initial value problem (2.13) and (2.14) when $\delta \to +0$ and derive the shallow water equations, whose solution approximates $(\eta^\delta, \phi^\delta)$ in a suitable sense. Then, we will give a theorem which ensures a rigorous approximation of the water wave by the shallow water equations.

It follows from the first equation in (2.13) that

$$\phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 = O(\delta^2).$$

By (2.5) and (2.7),

$$\begin{aligned}(3.1)\qquad (\partial_{n+1}\Phi)(x, x_{n+1}, t) &= (\partial_{n+1}\Phi)(x, b(x), t) + \int_{b(x)}^{x_{n+1}} (\partial_{n+1}^2\Phi)(x, y, t)\mathrm{d}y \\ &= \delta^2\nabla b(x)\cdot\nabla\Phi(x, b(x), t) - \delta^2\int_{b(x)}^{x_{n+1}} (\Delta\Phi)(x, y, t)\mathrm{d}y,\end{aligned}$$

which implies that $(\partial_{n+1}\Phi)(X, t) = O(\delta^2)$. Therefore,

$$\begin{aligned}\nabla\Phi(x, x_{n+1}, t) &= \nabla\Phi(x, 1+\eta(x,t), t) + \int_{1+\eta(x,t)}^{x_{n+1}} (\nabla\partial_{n+1}\Phi)(x, y, t)\mathrm{d}y \\ &= \nabla\Phi(x, 1+\eta(x,t), t) + O(\delta^2).\end{aligned}$$

Moreover, by the definition (2.9) it holds that

$$\begin{aligned}\nabla\phi(x,t) &= \nabla\Phi(x, 1+\eta(x,t), t) + \nabla\eta(x)(\partial_{n+1}\Phi)(x, 1+\eta(x), t) \\ &= \nabla\Phi(x, 1+\eta(x,t), t) + O(\delta^2) \\ &= \nabla\Phi(X,t) + O(\delta^2).\end{aligned}$$

Similarly, we have

$$\Delta\phi(x,t) = \Delta\Phi(X,t) + O(\delta^2).$$

These relation and (3.1) imply that

$$\begin{aligned}(\partial_{n+1}\Phi)(x, 1+\eta(x,t), t) &= \delta^2\nabla b(x)\cdot\nabla\phi(x,t) - \delta^2\int_{b(x)}^{1+\eta(x,t)} \Delta\phi(x,t)\mathrm{d}y + O(\delta^4) \\ &= -\delta^2(1+\eta(x,t))\Delta\phi(x,t) + \delta^2\nabla\cdot\big(b(x)\nabla\phi(x,t)\big) + O(\delta^4).\end{aligned}$$

Hence, by (2.11) we have

$$(3.2)\qquad (\Lambda\phi)(x,t) = -\nabla\cdot\big((1+\eta(x,t)-b(x))\nabla\phi(x,t)\big) + O(\delta^2).$$

This formal expansion of the operator $\Lambda = \Lambda(\eta, b, \delta)$ with respect to δ^2 can be justified mathematically by the following lemma.

Lemma 3.1 ([4]). *Let $M, c > 0$ and $s > n/2$. There exist positive constants C and δ_1 such that for any $\delta \in (0, \delta_1]$ and $\eta, b \in H^{s+2+1/2}(\mathbf{R}^n)$ satisfying*

$$\|b\|_{s+2+1/2} + \|\eta\|_{s+2+1/2} \le M, \qquad 1+\eta(x)-b(x) \ge c \quad \textit{for} \quad x \in \mathbf{R}^n,$$

we have

$$\|\Lambda(\eta, b, \delta)\phi + \nabla\cdot((1+\eta-b)\nabla\phi)\|_s \le C\delta^2\|\nabla\phi\|_{s+2}.$$

The second equation in (2.13) and (3.2) imply that

$$\eta_t + \nabla \cdot \big((1+\eta-b)\nabla\phi\big) = O(\delta^2).$$

To summarize, we have derived the partial differential equations

$$\begin{cases} \eta_t + \nabla \cdot \big((1+\eta-b)\nabla\phi\big) = O(\delta^2), \\ \phi_t + \eta + \frac{1}{2}|\nabla\phi|^2 = O(\delta^2), \end{cases}$$

which approximate the equations in (2.13) up to order δ^2. Letting $\delta \to 0$ in the above equations we obtain

$$\begin{cases} \eta^0_t + \nabla \cdot \big((1+\eta^0-b)\nabla\phi^0\big) = 0, \\ \phi^0_t + \eta^0 + \frac{1}{2}|\nabla\phi^0|^2 = 0. \end{cases}$$

Finally, putting $u^0 := \nabla\phi^0$ and taking the gradient of the second equation, we are led to the shallow water equations

$$\begin{cases} \eta^0_t + \nabla \cdot \big((1+\eta^0-b)u^0\big) = 0, \\ u^0_t + (u^0\cdot\nabla)u^0 + \nabla\eta^0 = 0. \end{cases} \tag{3.3}$$

Moreover, u^0 satisfies the irrotational condition

$$\operatorname{rot} u^0 = 0, \tag{3.4}$$

where $\operatorname{rot} u$ is the rotation of $u = (u_1, \dots, u_n)^T$ defined by $\operatorname{rot} u = (\partial_j u_i - \partial_i u_j)_{1\le i,j\le n}$.

The following theorem gives a mathematically rigorous justification of the shallow water equations for water waves.

Theorem 3.1 ([4]). *In addition to hypothesis of Theorem 2.1 we assume that as $\delta \to +0$ the initial data $(\eta^\delta_0, \nabla\phi^\delta_0)$ converge to (η^0_0, u^0_0) in $H^{s+3} \times H^{s+2}$. Then, as $\delta \to +0$ the solution obtained in Theorem 2.1 satisfies*

$$(\eta^\delta, \nabla\phi^\delta) \to (\eta^0, u^0) \quad \text{weakly* in} \quad L^\infty(0,T; H^{s+3}\times H^{s+2}),$$
$$\text{strongly in} \quad C([0,T]; H^{s+3-\epsilon}\times H^{s+2-\epsilon})$$

for each $\epsilon > 0$, where (η^0, u^0) is a unique solution of the shallow water equations (3.3) with initial conditions $(\eta^0, u^0)|_{t=0} = (\eta^0_0, u^0_0)$ and u^0 satisfies the irrotational condition (3.4).

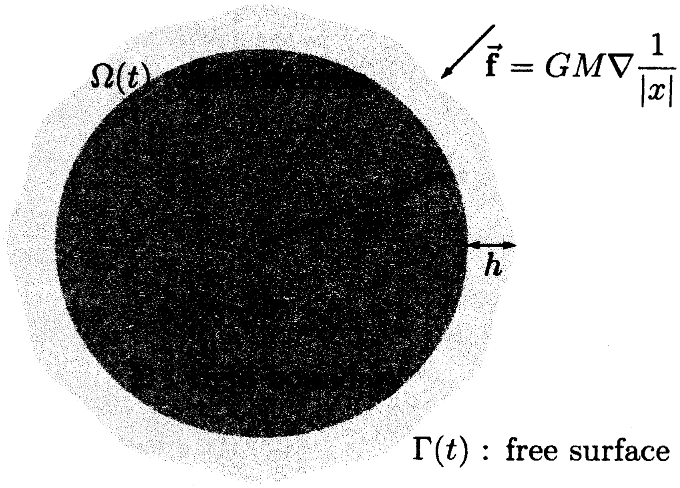
Moreover, if we also assume that $\|\eta^\delta_0 - \eta^0_0\|_s + \|\nabla\phi^\delta_0 - u^0_0\|_s = O(\delta^2)$, then for any $\delta \in (0, \delta_0]$ and $t \in [0,T]$ we have

$$\|\eta^\delta(t) - \eta^0(t)\|_s + \|\nabla\phi^\delta(t) - u^0(t)\|_s \le C\delta^2$$

with a constant C independent of δ and t.

4 Formulation of Problem II

The second type of the water wave corresponds to the ocean around the earth, that is, we take an effect of the curvature into account on the surface of the earth, and the shape of the fluid region is shown in the following illustration.



More precisely, we will consider a water wave around a 3-dimensional obstacle subject to the gravitation due to the obstacle. In this case, it would be better to use the radial coordinate r and the spherical coordinates ω, which moves on the unit sphere $\mathbf{S}^2$, rather than the Cartesian coordinates. We assume that the domain $\Omega(t)$ occupied by the fluid at time $t \geq 0$, the free surface $\Gamma(t)$, and the rigid boundary Σ of an obstacle are of the forms

$$\begin{aligned}
\Omega(t) &= \{x = r\omega \in \mathbf{R}^3\,;\, R + b(\omega) < r < R + h + \eta(\omega,t),\ \omega \in \mathbf{S}^2\},\\
\Gamma(t) &= \{x = r\omega \in \mathbf{R}^3\,;\, r = R + h + \eta(\omega,t),\ \omega \in \mathbf{S}^2\},\\
\Sigma &= \{x = r\omega \in \mathbf{R}^3\,;\, r = R + b(\omega),\ \omega \in \mathbf{S}^2\},
\end{aligned}$$

where R and h are the mean radius of the obstacle and the mean depth of the fluid, respectively. The functions b and η represent the bottom topography and the surface elevation, respectively. In this problem b is a given function, while η is the unknown.

We assume that the fluid is incompressible and inviscid, and that the flow is irrotational. Then, the fluid motion is described by the velocity potential $\Phi = \Phi(r,\omega,t)$ satisfying Laplace's equation in the spherical polar coordinates

$$(4.1) \qquad (r^2\Phi_r)_r + \Delta_{\mathbf{S}^2}\Phi = 0 \quad \text{in} \quad \Omega(t), \quad t > 0,$$

where $\Delta_{\mathbf{S}^2}$ is the Laplace-Beltrami operator on the unit sphere $\mathbf{S}^2$. The boundary conditions on the free surface are given by

$$(4.2) \qquad \begin{cases} \eta_t + \frac{1}{r^2}\nabla_{\mathbf{S}^2}\Phi\cdot\nabla_{\mathbf{S}^2}\eta - \Phi_r = 0, \\ \Phi_t + \frac{1}{2}\left(\Phi_r^2 + \frac{1}{r^2}|\nabla_{\mathbf{S}^2}\Phi|^2\right) - MG\left(\frac{1}{r} - \frac{1}{R+h}\right) = 0 \quad \text{on} \quad \Gamma(t), \quad t > 0, \end{cases}$$

where M is the total mass of the obstacle and G is the gravitational constant. It is assumed that the center of the gravity is located at the origin of coordinates. The gradient of a scalar field f and the divergence of a vector field u are denoted by $\nabla_{\mathbf{S}^2} f$ and $\nabla_{\mathbf{S}^2} \cdot u$, respectively. The first equation is the kinematical condition and the second one is what is known as Bernoulli's law restricted on the free surface. The boundary condition on the bottom is given by

$$\Phi_r - \tfrac{1}{r^2}\nabla_{\mathbf{S}^2}\Phi \cdot \nabla_{\mathbf{S}^2} b = 0 \qquad \text{on} \quad \Sigma, \quad t > 0. \tag{4.3}$$

Finally, we impose the initial conditions

$$\eta(\omega, 0) = \eta_0(\omega), \quad \Phi(r, \omega, 0) = \Phi_0(r, \omega). \tag{4.4}$$

It should be assumed that the initial data satisfy the compatibility conditions, that is, $(r^2\Phi_{0r})_r + \Delta_{\mathbf{S}^2}\Phi_0 = 0$ in $\Omega(0)$ and $\frac{1}{r^2}\nabla_{\mathbf{S}^2}\Phi_0 \cdot \nabla_{\mathbf{S}^2} b - \Phi_{0r} = 0$ on Σ.

We proceed to rewrite the equations (4.1)–(4.4) in an appropriate non-dimensional form. In this type of the water wave, a non-dimensional parameter δ that represents the shallowness of the water is defined by

$$\delta := \frac{h}{R}.$$

We rescale the independent and dependent variables by

$$r = R\tilde{r}, \quad t = \frac{R^2}{\sqrt{MGh(1+\delta)^{-1}}}\tilde{t}, \quad \Phi = \sqrt{MGh(1+\delta)^{-1}}\tilde{\Phi}, \quad \eta = h\tilde{\eta}, \quad b = h\tilde{b}.$$

Putting these into (4.1)–(4.4) and dropping the tilde sign in the notation we obtain

$$(r^2\Phi_r)_r + \Delta_{\mathbf{S}^2}\Phi = 0 \qquad \text{in} \quad \Omega(t), \quad t > 0. \tag{4.5}$$

$$\begin{cases} \delta\big(\eta_t + r^{-2}\nabla_{\mathbf{S}^2}\Phi \cdot \nabla_{\mathbf{S}^2}\eta\big) - \Phi_r = 0, \\ \Phi_t + \frac{1}{2}\big(\Phi_r^2 + r^{-2}|\nabla_{\mathbf{S}^2}\Phi|^2\big) + r^{-1}\eta = 0 \qquad \text{on} \quad \Gamma(t), \quad t > 0, \end{cases} \tag{4.6}$$

$$\Phi_r - \delta r^{-2}\nabla_{\mathbf{S}^2}\Phi \cdot \nabla_{\mathbf{S}^2} b = 0 \qquad \text{on} \quad \Sigma, \quad t > 0. \tag{4.7}$$

$$\eta(\omega, 0) = \eta_0^\delta(\omega), \quad \Phi(r, \omega, 0) = \Phi_0^\delta(r, \omega), \tag{4.8}$$

where

$$\begin{aligned} \Omega(t) &= \big\{x = r\omega \in \mathbf{R}^3\,;\, 1 + \delta b(\omega) < r < 1 + \delta\big(1 + \eta(\omega, t)\big),\ \omega \in \mathbf{S}^2\big\}, \\ \Gamma(t) &= \big\{x = r\omega \in \mathbf{R}^3\,;\, r = 1 + \delta\big(1 + \eta(\omega, t)\big),\ \omega \in \mathbf{S}^2\big\}, \\ \Sigma &= \big\{x = r\omega \in \mathbf{R}^3\,;\, r = 1 + \delta b(\omega),\ \omega \in \mathbf{S}^2\big\}. \end{aligned}$$

Since we are interested in asymptotic behavior of the solution when $\delta \to +0$, we always assume $0 < \delta \le 1$ in the following.

As before, we transform equivalently the initial value problem (4.5)–(4.8) to a problem on the free surface. To this end, we introduce new unknown function ϕ by

$$\phi(\omega, t) := \Phi\big(1 + \delta(1 + \eta(\omega, t)), \omega, t\big), \tag{4.9}$$

which is the trace of the velocity potential on the free surface. Then, we see that

$$\begin{cases} \phi_t = \Phi_t|_{\Gamma(t)} + \Phi_r|_{\Gamma(t)}\delta\eta_t, \\ \nabla_{\mathbf{S}^2}\phi = \nabla_{\mathbf{S}^2}\Phi|_{\Gamma(t)} + \Phi_r|_{\Gamma(t)}\delta\nabla_{\mathbf{S}^2}\eta. \end{cases} \tag{4.10}$$

It follows from (4.5), (4.7), and (4.9) that

$$\Lambda(\eta, b, \delta)\phi = \delta^{-1}r^2\big(\Phi_r - \delta r^{-2}\nabla_{\mathbf{S}^2}\eta \cdot \nabla_{\mathbf{S}^2}\Phi\big)\big|_{\Gamma(t)}, \tag{4.11}$$

where $\Lambda = \Lambda(\eta, b, \delta)$ is a linear operator called the Dirichlet-to-Neumann map for Laplace's equation. In this case, the map $\Lambda = \Lambda(\eta, b, \delta)$ is defined as follows.

Definition 4.1. Under appropriate assumptions on η and b, for any function φ on the free surface in some class there exists a unique solution Φ of the boundary value problem

$$\begin{cases} (r^2\Phi_r)_r + \Delta_{\mathbf{S}^2}\Phi = 0 & \text{in} \quad 1 + \delta b(\omega) < r < 1 + \delta\big(1 + \eta(\omega, t)\big), \\ \Phi = \varphi & \text{on} \quad r = 1 + \delta\big(1 + \eta(\omega, t)\big), \\ \Phi_r - \delta r^{-2}\nabla_{\mathbf{S}^2}\Phi \cdot \nabla_{\mathbf{S}^2}b = 0 & \text{on} \quad r = 1 + \delta b(\omega). \end{cases}$$

Note that in the Cartesian coordinates this boundary value problem can be written in the form

$$\begin{cases} \Delta\Phi = 0 & \text{in} \quad \Omega(t), \\ \Phi = \varphi & \text{on} \quad \Gamma(t), \\ N \cdot \nabla\Phi = 0 & \text{on} \quad \Sigma. \end{cases}$$

Using the solution Φ we define the Dirichlet-to-Neumann map $\Lambda = \Lambda(\eta, b, \delta)$ by

$$\begin{aligned} \Lambda(\eta, b, \delta)\varphi &:= \delta^{-1}r^2\big(\Phi_r - \delta r^{-2}\nabla_{\mathbf{S}^2}\eta \cdot \nabla_{\mathbf{S}^2}\Phi\big)\big|_{\Gamma(t)} \\ &\big(= \delta^{-1}r^2\sqrt{1 + \delta^2 r^{-2}}N \cdot \nabla\Phi|_{\Gamma(t)}\ \big). \end{aligned}$$

The second equation in (4.10) and (4.11) imply that

$$\begin{cases} \Phi_r|_{\Gamma(t)} = \delta\big(r^2 + \delta^2|\nabla_{\mathbf{S}^2}\eta|^2\big)^{-1}\big(\Lambda\phi + \nabla_{\mathbf{S}^2}\eta \cdot \nabla_{\mathbf{S}^2}\phi\big), \\ \nabla_{\mathbf{S}^2}\Phi|_{\Gamma(t)} = \nabla_{\mathbf{S}^2}\phi - \delta^2\big(r^2 + \delta^2|\nabla_{\mathbf{S}^2}\eta|^2\big)^{-1}\big(\Lambda\phi + \nabla_{\mathbf{S}^2}\eta \cdot \nabla_{\mathbf{S}^2}\phi\big)\nabla_{\mathbf{S}^2}\eta. \end{cases} \tag{4.12}$$

It follows from the first equation in (4.6) and (4.11) that $\eta_t - r^{-2}\Lambda\phi = 0$, so that by the first equation in (4.10) we get

$$\Phi_t|_{\Gamma(t)} = \phi_t - \delta^2 r^{-2}\big(r^2 + \delta^2|\nabla_{\mathbf{S}^2}\eta|^2\big)^{-1}\big(\Lambda\phi + \nabla_{\mathbf{S}^2}\eta \cdot \nabla_{\mathbf{S}^2}\phi\big)\Lambda\phi.$$

Putting this and (4.12) into the second equation in (4.6) we obtain

$$\text{(4.13)} \qquad \begin{cases} \phi_t + r^{-1}\eta + \frac{1}{2}r^{-2}|\nabla_{\mathbf{S}^2}\phi|^2 \\ \quad -\frac{1}{2}\delta^2 r^{-2}\left(r^2 + \delta^2|\nabla_{\mathbf{S}^2}\eta|^2\right)^{-1}\left(\Lambda(\eta, b, \delta)\phi + \nabla_{\mathbf{S}^2}\eta \cdot \nabla_{\mathbf{S}^2}\phi\right)^2 = 0, \\ \eta_t - r^{-2}\Lambda(\eta, b, \delta)\phi = 0 \qquad \text{for} \quad t > 0, \end{cases}$$

$$\text{(4.14)} \qquad \eta = \eta_0^\delta, \quad \phi = \phi_0^\delta \qquad \text{at} \quad t = 0,$$

where $r = 1 + \delta(1 + \eta)$ and $\phi_0^\delta = \Phi_0^\delta\big(1 + \delta(1 + \eta_0^\delta(\cdot)), \cdot\big)$. This is another initial value problem that we are going to investigate in this communication.

5 Shallow water approximation for Problem II

We proceed to study formally asymptotic behavior of the solution $(\eta^\delta, \phi^\delta)$ to the initial value problem (4.13) and (4.14) when $\delta \to +0$ and derive the shallow water equations on the sphere $\mathbf{S}^2$, whose solution approximates $(\eta^\delta, \phi^\delta)$ in a suitable sense.

It follows from the first equation in (4.13) that

$$\phi_t + \eta + \frac{1}{2}|\nabla_{\mathbf{S}^2}\phi|^2 = O(\delta).$$

By (4.7),

$$\text{(5.1)} \qquad \begin{aligned} \Phi_r(r, \omega, t) &= \Phi_r\big|_{r=1+\delta b(\omega)} + \int_{1+\delta b(\omega)}^{r} \Phi_{rr}(s, \omega, t)\mathrm{d}s \\ &= \delta r^{-2}\nabla_{\mathbf{S}^2}\Phi\big|_{r=1+\delta b(\omega)} \cdot \nabla_{\mathbf{S}^2} b + \int_{1+\delta b(\omega)}^{r} \Phi_{rr}(s, \omega, t)\mathrm{d}s. \end{aligned}$$

Since $1 + \delta b(\omega) < r < 1 + \delta(1 + \eta(\omega, t))$, (5.1) implies that $\Phi_r(r, \omega, t) = O(\delta)$. Therefore,

$$\Phi(r, \omega, t) = \phi(\omega, t) + \int_{1+\delta(1+\eta(\omega,t))}^{r} \Phi_r(s, \omega, t)\mathrm{d}s = \phi(\omega, t) + O(\delta),$$

so that by (4.5),

$$\Phi_{rr}(r, \omega, t) = -2r^{-1}\Phi_r(r, \omega, t) - r^{-2}\Delta_{\mathbf{S}^2}\Phi(r, \omega, t) = -\Delta_{\mathbf{S}^2}\phi(\omega, t) + O(\delta).$$

Putting these into (5.1) we see that

$$\Phi_r\big|_{r=1+\delta(1+\eta(\omega,t))} = \delta\nabla_{\mathbf{S}^2}\eta \cdot \nabla_{\mathbf{S}^2} b - \delta(1 + \eta - b)\Delta_{\mathbf{S}^2}\phi + O(\delta^2).$$

Hence, by (4.11) we have

$$\text{(5.2)} \qquad \Lambda\phi = -\nabla_{\mathbf{S}^2} \cdot \big((1 + \eta - b)\nabla_{\mathbf{S}^2}\phi\big) + O(\delta).$$

This and the second equation in (4.13) imply that

$$\eta_t + \nabla_{\mathbf{S}^2} \cdot \big((1+\eta-b)\nabla_{\mathbf{S}^2}\phi\big) = O(\delta).$$

To summarize, we have derived the partial differential equations

$$\begin{cases} \eta_t + \nabla_{\mathbf{S}^2} \cdot \big((1+\eta-b)\nabla_{\mathbf{S}^2}\phi\big) = O(\delta), \\ \phi_t + \eta + \frac{1}{2}|\nabla_{\mathbf{S}^2}\phi|^2 = O(\delta), \end{cases}$$

which approximate the equations in (4.13) up to order δ. Letting $\delta \to 0$ in the above equations we obtain

$$\begin{cases} \eta^0_t + \nabla_{\mathbf{S}^2} \cdot \big((1+\eta^0-b)\nabla_{\mathbf{S}^2}\phi^0\big) = 0, \\ \phi^0_t + \eta^0 + \frac{1}{2}|\nabla_{\mathbf{S}^2}\phi^0|^2 = 0. \end{cases}$$

Finally, putting $u^0 := \nabla_{\mathbf{S}^2}\phi^0$ and taking the gradient of the second equation, we are led to the shallow water equations on the sphere $\mathbf{S}^2$

$$\begin{cases} \eta^0_t + \nabla_{\mathbf{S}^2} \cdot \big((1+\eta^0-b)u^0\big) = 0, \\ u^0_t + \nabla_{u^0}u^0 + \nabla_{\mathbf{S}^2}\eta^0 = 0, \end{cases} \tag{5.3}$$

where $\nabla_{u^0}u^0$ is the covariant derivative of the vector field u^0 with respect to u^0. These have exactly the same form as the compressible Euler equations on the manifold $\mathbf{S}^2$, so that this shallow water limit gives the necessity to the analysis of the compressible Euler equations not only in the Euclidean space but also on general manifolds.

6 Linearized equations and energy estimates

The most difficult part to give a mathematically rigorous justification of the shallow water approximations for water waves is to establish an existence theory for the initial value problems (2.13) and (2.14), and (4.13) and (4.14) together with uniform boundedness of the solution with respect to the small parameter δ. Such uniform boundedness are obtained by the energy methods together with a precise analysis of the Dirichlet-to-Neumann map Λ for Laplace's equation. In the analysis, we transform the boundary value problem for Laplace's equation in the fluid domain $\Omega(t)$ to a problem on the simple fixed domain $\Omega_0 = \mathbf{R}^n \times (0,1)$ in the case of Problem I and $\Omega_0 = \{x = r\omega \in \mathbf{R}^3\,;\, 1 < r < 1+\delta,\ \omega \in \mathbf{S}^2\}$ in the case of Problem II, respectively, by using an appropriate diffeomorphism $\Theta : \Omega_0 \to \Omega(t)$. This is one of the crucial parts of this communication. We will construct such a diffeomorphism Θ which is conformal in the tangential and the normal directions on the boundary in some sense.

In order to explain how to apply the method to our problem, we will focus on the initial value problem (2.13) and (2.14) and consider linearized equations of (2.13) around an arbitrary flow (η, ϕ) and give an energy estimate of the solution to the linearized equations. The energy estimate for the problem (4.13) and (4.14) can be carried out in almost the same way. Following D. Lannes [8], we linearize the equations in (2.13) around (η, ϕ). To this end, we need to calculate the Fréchet derivative of the Dirichlet-to-Neumann map $\Lambda(\eta, b, \delta)$ with respect to η.

Lemma 6.1 ([8]). *The Fréchet derivative of $\Lambda(\eta, b, \delta)$ with respect to η has the form*

$$D_\eta \Lambda(\eta, b, \delta)[\zeta]\phi = -\delta^2 \Lambda(\eta, b, \delta)(Z\zeta) - \nabla \cdot (v\zeta),$$

where

$$\begin{cases} Z = (1 + \delta^2 |\nabla \eta|^2)^{-1}(\Lambda(\eta, b, \delta)\phi + \nabla\eta \cdot \nabla\phi), \\ v = \nabla\phi - \delta^2 Z \nabla\eta. \end{cases}$$

By this lemma, setting

$$\zeta := \partial\eta, \qquad \psi := \partial\phi - \delta^2 Z \partial\eta,$$

we see that the linearized equations have the form

$$\begin{cases} \psi_t + v \cdot \nabla\psi + (1 + \delta^2 Z_t + \delta^2 v \cdot \nabla Z)\zeta = 0, \\ \zeta_t + \nabla \cdot (v\zeta) - \Lambda\psi = D_b \Lambda[\partial b]\phi. \end{cases}$$

Here, we note that the function $1 + \delta^2 Z_t + \delta^2 v \cdot \nabla Z$ is positively definite for sufficiently small δ. In view of this, we will consider the following system of linear equations for unknowns (ψ, ζ).

$$\begin{cases} \psi_t + b_1 \cdot \nabla\psi + a\zeta = f_1, \\ \zeta_t + b_2 \cdot \nabla\zeta - \Lambda\psi = f_2, \end{cases} \tag{6.1}$$

where a, $b_1 = (b_{11}, \ldots, b_{1n})$, $b_2 = (b_{21}, \ldots, b_{2n})$, f_1, f_2 are given functions of x and t and may depend on δ, and $\Lambda = \Lambda(\eta, b, \delta)$ is the Dirichlet-to-Neumann map. We assume that the function a satisfies the following positivity condition.

$$a(x,t) \geq c_0 > 0 \qquad \text{for} \quad x \in \mathbf{R}^n,\ 0 \leq t \leq T.$$

In order to define an energy function to the system (6.1), we need more information on the Dirichlet-to-Neumann map Λ.

Introducing a $(n+1) \times (n+1)$ matrix I_δ by

$$I_\delta = \begin{pmatrix} E_n & 0 \\ 0 & \delta^{-1} \end{pmatrix},$$

where E_n is the $n \times n$ unit matrix, we can rewrite the boundary value problem in Definition 2.1 as the following form.

$$\begin{cases} \nabla_X \cdot I_\delta^2 \nabla_X \Phi = 0 & \text{in} \quad \Omega, \\ \Phi = \phi & \text{on} \quad \Gamma, \\ N \cdot I_\delta^2 \nabla_X \Phi = 0 & \text{on} \quad \Sigma. \end{cases} \tag{6.2}$$

Lemma 6.2. *The Dirichlet-to-Neumann map* $\Lambda = \Lambda(\eta, b, \delta)$ *is symmetric in* $L^2(\mathbf{R}^n)$, *that is, for any* $\phi, \psi \in H^1(\mathbf{R}^n)$ *it holds that*

$$(\Lambda\phi, \psi) = (\phi, \Lambda\psi).$$

Proof. Set $\Phi := \phi^{\hbar}$ and $\Psi := \psi^{\hbar}$. By Green's formula we have

$$\begin{aligned} 0 &= \int_\Omega \big((\nabla_X \cdot I_\delta^2 \nabla_X \Phi)\Psi - \Phi(\nabla_X \cdot I_\delta^2 \nabla_X \Psi)\big)\mathrm{d}X \\ &= \int_\Gamma \big((N \cdot I_\delta^2 \nabla_X \Phi)\Psi - \Phi(N \cdot I_\delta^2 \nabla_X \Psi)\big)\mathrm{d}S, \end{aligned}$$

where N is the unit outward normal to the boundary $\partial\Omega$. In the above calculation we used the boundary condition on the bottom Σ. Since $\Phi = \phi$, $\Psi = \psi$, $\sqrt{1+|\nabla\eta|^2}N \cdot I_\delta^2 \nabla_X \Phi = \Lambda\phi$, $\sqrt{1+|\nabla\eta|^2}N \cdot I_\delta^2 \nabla_X \Psi = \Lambda\psi$, and $\mathrm{d}S = \sqrt{1+|\nabla\eta|^2}\mathrm{d}x$ on Γ, we obtain the desired identity. □

Lemma 6.3. *For any* $\phi \in H^1(\mathbf{R}^n)$, *it holds that* $(\Lambda\phi, \phi) = \|I_\delta \nabla_X \Phi\|^2_{L^2(\Omega)}$, *where* $\Phi = \phi^{\hbar}$.

Proof. By Green's formula we see that

$$0 = \int_\Omega (\nabla_X \cdot I_\delta^2 \nabla_X \Phi)\Phi \mathrm{d}X = \int_{\partial\Omega} (N \cdot I_\delta^2 \nabla_X \Phi)\Phi \mathrm{d}S - \int_\Omega |I_\delta \nabla_X \Phi|^2 \mathrm{d}X.$$

This together with the boundary conditions yields the desired identity. □

These two lemmas imply that the Dirichlet-to-Neumann map Λ is a positive operator in $L^2(\mathbf{R}^n)$. For simplicity, we first consider the linear equations (6.1) in the case $b_1 = b_2 = 0$, that is, the equations

$$\begin{cases} \psi_t + a\zeta = f_1, \\ \zeta_t - \Lambda\psi = f_2, \end{cases}$$

which can be written in the matrix form

$$\begin{pmatrix} \psi \\ \zeta \end{pmatrix}_t + \begin{pmatrix} 0 & a \\ -\Lambda & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \zeta \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

or

$$\mathscr{A}_0 U_t + \mathscr{A}_1 U = F$$

where $U = (\psi, \zeta)^T$, $F = (\Lambda f_1, a f_2)^T$ and

$$\mathscr{A}_0 = \begin{pmatrix} \Lambda & 0 \\ 0 & a \end{pmatrix}, \qquad \mathscr{A}_1 = \begin{pmatrix} 0 & \Lambda a \\ -a\Lambda & 0 \end{pmatrix}.$$

Here, we note that $\mathscr{A}_0$ is positively definite and $\mathscr{A}_1$ is skew-symmetric, that is, $\mathscr{A}_1^* = -\mathscr{A}_1$. This means that the matrix operator $\mathscr{A}_0$ is a symmetrizer for the system (6.1), so that the corresponding energy function is defined by

$$E(t) := (\mathscr{A}_0 U, U) = (\Lambda\psi, \psi) + (a\zeta, \zeta).$$

In fact, for any smooth solution (ψ, ζ) to the system (6.1) we see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= ([\partial_t, \Lambda]\psi, \psi) + 2(\psi_t, \Lambda\psi) + (a_t\zeta, \zeta) + 2(a\zeta_t, \zeta)\\ &= ([\partial_t, \Lambda]\psi, \psi) - 2(b_1 \cdot \nabla\psi, \Lambda\psi) + 2(f_1, \Lambda\psi)\\ &\quad + (a_t\zeta, \zeta) + ((\nabla \cdot (ab_2))\zeta, \zeta) + 2(af_2, \zeta).\end{aligned}$$

Crucial terms in the right hand side are $([\partial_t, \Lambda]\psi, \psi)$ and $(b_1 \cdot \nabla\psi, \Lambda\psi)$.

Lemma 6.4. *Let $r > n/2$, $c_0, M > 0$. There exist positive constants C_1 and δ_1 such that if $0 < \delta \le \delta_1$, $b \in H^{r+1}$ and $\eta \in C^1([0,T]; H^{r+1})$ satisfy the conditions*

$$\begin{cases} \|b\|_{r+1} + \|\eta(t)\|_{r+1} + \|\eta_t(t)\|_{r+1} \le M,\\ 1 + \eta(x,t) - b(x) \ge c_0 \quad \text{for} \quad x \in \mathbf{R}^n, \quad 0 \le t \le T,\end{cases}$$

then we have

$$|([\partial_t, \Lambda]\phi, \phi)| \le C_1(\Lambda\phi, \phi).$$

Proof. Taking an appropriate diffeomorphism $\Theta : \Omega_0 = \mathbf{R}^n \times [0,1] \to \overline{\Omega(t)}$, we put $\Phi := \phi^h$ and $\tilde{\Phi} := \Phi \circ \Theta$. Then, the boundary value problem (6.2) is transformed into

$$\begin{cases} \nabla_X \cdot I_\delta P I_\delta \nabla_X \tilde{\Phi} = 0 & \text{in} \quad 0 < x_{n+1} < 1,\\ \tilde{\Phi} = \phi & \text{on} \quad x_{n+1} = 1,\\ \partial_{n+1}\tilde{\Phi} = 0 & \text{on} \quad x_{n+1} = 0,\end{cases}$$

where $P = P(x, y, t; \delta)$ is positively definite and satisfies

$$\begin{cases} |P| + |P^{-1}| + |P_t| \le C,\\ P(x,0) = \begin{pmatrix} * & 0\\ 0 & 1\end{pmatrix}, \quad P(x,1) = \begin{pmatrix} * & 0\\ 0 & 1\end{pmatrix}.\end{cases}$$

Moreover, it holds that

$$C^{-1}\|I_\delta \nabla_X \Phi\|_{L^2(\Omega)} \le \|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \le C\|I_\delta \nabla_X \Phi\|_{L^2(\Omega)}. \tag{6.3}$$

In fact, we can construct such a diffeomorphism Θ if we take δ_1 sufficiently small. Then, by Lemma 6.3 we have

$$(\Lambda\phi, \phi) = \int_{\Omega(t)} |I_\delta \nabla_X \Phi|^2 \mathrm{d}X = \int_{\Omega_0} P I_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Phi} \mathrm{d}X,$$

so that

$$([\partial_t, \Lambda]\phi, \phi) = \frac{\mathrm{d}}{\mathrm{d}t}(\Lambda\phi, \phi) = 2\int_{\Omega_0} PI_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Phi}_t \mathrm{d}X + \int_{\Omega_0} P_t I_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Phi} \mathrm{d}X.$$

Since $\tilde{\Phi}(\cdot, 1) = \phi$, we have $\tilde{\Phi}_t(\cdot, 1) = 0$. Therefore, by Green's formula we see that

$$\begin{aligned}
&\int_{\Omega_0} PI_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Phi}_t \mathrm{d}X \\
&= -\int_{\Omega_0} (\nabla_X \cdot I_\delta P I_\delta \nabla_X \tilde{\Phi})\tilde{\Phi}_t \mathrm{d}X \\
&\quad + (\mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X \tilde{\Phi}(\cdot, 1), \tilde{\Phi}_t(\cdot, 1)) - (\mathbf{e}_{n+1} \cdot I_\delta^2 \nabla_X \tilde{\Phi}(\cdot, 0), \tilde{\Phi}_t(\cdot, 0)) \\
&= 0.
\end{aligned}$$

Hence, we obtain

$$|([\partial_t, \Lambda]\phi, \phi)| \le \|P_t\|_{L^\infty(\Omega_0)} \|I_\delta \nabla_X \tilde{\Phi}\|^2_{L^2(\Omega_0)}.$$

This together with (6.3) and Lemma 6.3 implies the desired estimate. □

Lemma 6.5. *Let $r > n/2$, $c_0, M > 0$. There exist positive constants C_1 and δ_1 such that if $0 < \delta \le \delta_1$, $b, \eta \in H^{r+2}$satisfy the conditions*

$$\begin{cases} \|b\|_{r+2} + \|\eta\|_{r+2} \le M, \\ 1 + \eta(x) - b(x) \ge c_0 \quad \text{for} \quad x \in \mathbf{R}^n, \end{cases}$$

then we have

$$|(\Lambda\phi, v \cdot \nabla\phi)| \le C_1 \|v\|_{r+1} (\Lambda\phi, \phi).$$

Proof. We set $\Phi := \phi^\hbar$ and construct a vector field $V = (V_1, \dots, V_n, V_{n+1})^T$ on Ω satisfying

$$\begin{cases} V_j|_\Gamma = v_j \quad (1 \le j \le n), \qquad V_{n+1}|_\Gamma = \delta v \cdot \nabla\eta, \\ V_{n+1}|_\Sigma = \delta(V_1|_\Sigma, \dots, V_n|_\Sigma)^T \cdot \nabla b, \end{cases}$$

and

$$\|I_\delta \nabla_X V_1\|_{L^\infty(\Omega)} + \cdots + \|I_\delta \nabla_X V_{n+1}\|_{L^\infty(\Omega)} \le C\|v\|_{r+1}. \tag{6.4}$$

Then, it is easy to see that

$$V \cdot I_\delta \nabla_X \Phi\big|_\Gamma = v \cdot \nabla\phi, \qquad V \cdot I_\delta N\big|_\Gamma = V \cdot I_\delta N\big|_\Sigma = 0.$$

By these relations and Green's formula we see that

$$\begin{aligned}
(\Lambda\phi, v \cdot \nabla\phi) &= \int_\Gamma (N \cdot I_\delta^2 \nabla_X \Phi)(V \cdot I_\delta \nabla_X \Phi) \mathrm{d}S = \int_\Omega \nabla_X \cdot \big((I_\delta^2 \nabla_X \Phi)(V \cdot I_\delta \nabla_X \Phi)\big) \mathrm{d}X \\
&= \int_\Omega I_\delta \nabla_X \Phi \cdot (I_\delta \nabla_X V) I_\delta \nabla_X \Phi \mathrm{d}X + \frac{1}{2}\int_\Omega V \cdot I_\delta \nabla_X |I_\delta \nabla_X \Phi|^2 \mathrm{d}X \\
&= \int_\Omega \Big(I_\delta \nabla_X \Phi \cdot (I_\delta \nabla_X V) I_\delta \nabla_X \Phi - \frac{1}{2}(I_\delta \nabla_X \cdot V)|I_\delta \nabla_X \Phi|^2 \Big) \mathrm{d}X,
\end{aligned}$$

where $I_\delta \nabla_X V = (I_\delta \nabla_X V_1, \dots, I_\delta \nabla_X V_{n+1})$. Therefore, we obtain

$$|(\Lambda\phi, v \cdot \nabla\phi)| \le C\|I_\delta \nabla_X V\|_{L^\infty(\Omega)} \|I_\delta \nabla_X \Phi\|^2_{L^2(\Omega)} = C\|I_\delta \nabla_X V\|_{L^\infty(\Omega)} (\Lambda\phi, \phi),$$

which together with (6.4) implies the desired estimate. □

Lemma 6.6. *For the Dirichlet-to-Neumann map $\Lambda = \Lambda(\eta, b, \delta)$ it holds that*

$$|(\phi, \Lambda\psi)| \le \sqrt{(\phi, \Lambda\phi)}\sqrt{(\psi, \Lambda\psi)}.$$

Proof. Set $\Phi := \phi^h$ and $\Psi := \psi^h$. By Green's formula we see that

$$(\Lambda\phi, \psi) = \int_\Gamma (N \cdot I_\delta^2 \nabla_X \Phi)\Psi \mathrm{d}S = \int_\Omega \nabla_X \cdot ((I_\delta^2 \nabla_X \Phi)\Psi)\mathrm{d}X = \int_\Omega I_\delta \nabla_X \Phi \cdot I_\delta \nabla_X \Psi \mathrm{d}X.$$

Therefore, by Lemma 6.3 we obtain

$$|(\Lambda\phi, \psi)| \le \|I_\delta \nabla_X \Phi\|_{L^2(\Omega)} \|I_\delta \nabla_X \Psi\|_{L^2(\Omega)} = \sqrt{(\phi, \Lambda\phi)}\sqrt{(\psi, \Lambda\psi)}.$$

This shows the desired estimate. □

By these Lemmas 6.4–6.6, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le CE(t) + \{(\Lambda f_1(t), f_1(t)) + \|f_2(t)\|^2\},$$

which together with Gronwall's inequality implies that

$$E(t) \le C\mathrm{e}^{Ct}E(0) + \int_0^t \mathrm{e}^{C(t-\tau)}\{(\Lambda f_1(\tau), f_1(\tau)) + \|f_2(\tau)\|^2\}\mathrm{d}\tau.$$

Similarly, for a high order energy function $E_s(t)$ defined by

$$E_s(t) := (\Lambda J^s\psi(t), J^s\psi(t)) + (aJ^s\zeta(t), J^s\zeta(t)),$$

where $J = 1 + |D|$ (we use the standard notation of Fourier multipliers), we can obtain a high order energy estimate

$$E_s(t) \le C\mathrm{e}^{Ct}E_s(0) + \int_0^t \mathrm{e}^{C(t-\tau)}\{(\Lambda J^s f_1(\tau), J^s f_1(\tau)) + \|f_2(\tau)\|_s^2\}\mathrm{d}\tau \tag{6.5}$$

with a constant C independent of δ.

Now, we need to convert the energy function $E_s(t)$ into the norm of a Sobolev space uniformly with respect to δ.

Lemma 6.7. *Under the same hypothesis of Lemma 6.4, for any $\phi \in H^1$ we have*

$$C^{-1}\|\Lambda_0^{1/2}\phi\|^2 \le (\Lambda\phi, \phi) \le C\|\Lambda_0^{1/2}\phi\|^2$$

with a constant $C \ge 1$ independent of δ, where $\Lambda_0 = \Lambda(0, 0, \delta) = \frac{1}{\delta}|D|\tanh(\delta|D|)$.

Proof. By using the diffeomorphism Θ in the proof of Lemma 6.4, we set $\Phi := \phi^h$ and $\tilde{\Phi} := \Phi \circ \Theta$, and decompose $\tilde{\Phi} = \tilde{\Phi}_1 + \tilde{\Phi}_2$, where $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are solutions of the boundary value problems

$$\begin{cases} \nabla_X \cdot I_\delta^2 \nabla_X \tilde{\Phi}_1 = 0 & \text{in} \quad 0 < x_{n+1} < 1, \\ \tilde{\Phi}_1 = \phi & \text{on} \quad x_{n+1} = 1, \\ \partial_{n+1} \tilde{\Phi}_1 = 0 & \text{on} \quad x_{n+1} = 0 \end{cases}$$

and

$$\begin{cases} \nabla_X \cdot I_\delta^2 \nabla_X \tilde{\Phi}_2 = \nabla_X \cdot I_\delta (I_1 - P) I_\delta \nabla_X \tilde{\Phi} & \text{in} \quad 0 < x_{n+1} < 1, \\ \tilde{\Phi}_2 = 0 & \text{on} \quad x_{n+1} = 1, \\ \partial_{n+1} \tilde{\Phi}_2 = 0 & \text{on} \quad x_{n+1} = 0, \end{cases}$$

respectively. Then, it holds that

$$\Lambda\phi = \delta^{-2}\partial_{n+1}\tilde{\Phi}(\cdot, 1) = \delta^{-2}\partial_{n+1}\tilde{\Phi}_1(\cdot, 1) + \delta^{-2}\partial_{n+1}\tilde{\Phi}_2(\cdot, 1) = \Lambda_0\phi + \delta^{-2}\partial_{n+1}\tilde{\Phi}_2(\cdot, 1)$$

and, by Lemma 6.3, that

$$(\Lambda\phi, \phi) = \|I_\delta \nabla_X \Phi\|^2_{L^2(\Omega)}, \qquad \|\Lambda_0^{1/2}\phi\|^2 = (\Lambda_0\phi, \phi) = \|I_\delta \nabla_X \tilde{\Phi}_1\|^2_{L^2(\Omega_0)}.$$

By Green's formula we see that

$$\begin{aligned} \big(\delta^{-2}\partial_{n+1}\tilde{\Phi}_2(\cdot, 1), \phi\big) &= \big(\delta^{-2}\partial_{n+1}\tilde{\Phi}_2(\cdot, 1), \tilde{\Phi}_1(\cdot, 1)\big) \\ &= \int_{\Omega_0} I_\delta \nabla_X \tilde{\Phi}_2 \cdot I_\delta \nabla_X \tilde{\Phi}_1 \mathrm{d}X + \int_{\Omega_0} (\nabla_X \cdot I_\delta^2 \nabla_X \tilde{\Phi}_2) \tilde{\Phi}_1 \mathrm{d}X \\ &= \int_{\Omega_0} I_\delta \nabla_X \tilde{\Phi}_2 \cdot I_\delta \nabla_X \tilde{\Phi}_1 \mathrm{d}X + \int_{\Omega_0} \big(\nabla_X \cdot I_\delta (I_1 - P) I_\delta \nabla_X \tilde{\Phi}\big) \tilde{\Phi}_1 \mathrm{d}X \\ &= \int_{\Omega_0} I_\delta \nabla_X \tilde{\Phi}_2 \cdot I_\delta \nabla_X \tilde{\Phi}_1 \mathrm{d}X - \int_{\Omega_0} (I_1 - P) I_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Phi}_1 \mathrm{d}X. \end{aligned}$$

Therefore,

$$\big|\big(\delta^{-2}\partial_{n+1}\tilde{\Phi}_2(\cdot, 1), \phi\big)\big| \le C(\|I_\delta \nabla_X \tilde{\Phi}_2\|_{L^2(\Omega_0)} + \|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)})\|I_\delta \nabla_X \tilde{\Phi}_1\|_{L^2(\Omega_0)}.$$

Similarly, by the equations for $\tilde{\Phi}_2$ we see that

$$\begin{aligned} \|I_\delta \nabla_X \tilde{\Phi}_2\|^2_{L^2(\Omega_0)} &= -\int_{\Omega_0} (\nabla_X \cdot I_\delta^2 \nabla_X \tilde{\Phi}_2) \tilde{\Phi}_2 \mathrm{d}X = -\int_{\Omega_0} (\nabla_X \cdot I_\delta (I_1 - P) I_\delta \nabla_X \tilde{\Phi}) \tilde{\Phi}_2 \mathrm{d}X \\ &= \int_{\Omega_0} (I_1 - P) I_\delta \nabla_X \tilde{\Phi} \cdot I_\delta \nabla_X \tilde{\Phi}_2 \mathrm{d}X \le C\|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \|I_\delta \nabla_X \tilde{\Phi}_2\|_{L^2(\Omega_0)}, \end{aligned}$$

so that

$$\|I_\delta \nabla_X \tilde{\Phi}_2\|_{L^2(\Omega_0)} \le C\|I_\delta \nabla_X \tilde{\Phi}\|_{L^2(\Omega_0)} \le C\|I_\delta \nabla_X \Phi\|_{L^2(\Omega)},$$

where we used (6.3). Summarizing the above estimates we obtain

$$\big|(\Lambda\phi, \phi) - (\Lambda_0\phi, \phi)\big| \le C_1 \|I_\delta \nabla_X \Phi\|_{L^2(\Omega)} \|I_\delta \nabla_X \tilde{\Phi}_1\|_{L^2(\Omega_0)} \le C_1 \sqrt{(\Lambda\phi, \phi)} \sqrt{(\Lambda_0\phi, \phi)},$$

which easily yields the desired inequalities. □

Lemma 6.8. *For any real s, we have*

$$\begin{cases} \|\nabla\phi\|_s \le \sqrt{2(1+\delta)}\|\Lambda_0^{1/2}\phi\|_{s+1/2}, \\ \|\Lambda_0^{1/2}\phi\|_s \le \min\{\|\nabla\phi\|_s, \delta^{-1/2}\|\phi\|_{s+1/2}\}. \end{cases}$$

Proof. By the inequalities $(1+\sqrt{\alpha})^{-1}\alpha \le \sqrt{\alpha\tanh\alpha} \le \min\{\alpha, \sqrt{\alpha}\}$ for $\alpha \ge 0$, it holds that

$$(1+\sqrt{\delta|\xi|})^{-1}|\xi| \le \sqrt{\delta^{-1}|\xi|\tanh(\delta|\xi|)} \le \min\{|\xi|, \delta^{-1/2}|\xi|^{1/2}\} \quad \text{for} \quad \xi \in \mathbf{R}^n,\ \delta > 0,$$

which yields the desired estimates. □

It follows from (6.5) and Lemmas 6.7 and 6.8 that for any smooth solution (ψ, ζ) to the system (6.1) of linear equations we have

$$\begin{aligned} &\|\nabla\psi(t)\|_{s-1/2}^2 + \|\zeta(t)\|_s^2 \\ &\le Ce^{Ct}(\|\nabla\psi(0)\|_s^2 + \|\zeta(0)\|_s^2) + C\int_0^t e^{C(t-\tau)}(\|\nabla f_1(\tau)\|_s^2 + \|f_2(\tau)\|_s^2)\mathrm{d}\tau \end{aligned}$$

with a constant C independent of δ.

For the nonlinear problem (2.13), we reduce the problem to a system of quasilinear equations by introducing new functions $\zeta_{ijk} := \partial_{ijk}\eta$ and $\psi_{ijk} := \partial_{ijk}\phi - \delta^2 Z\partial_{ijk}\eta$, where $\partial_{ijk} = \partial_i\partial_j\partial_k$ and Z is given in Lemma 6.1. Then, the system has the form

$$\begin{cases} \partial_t\zeta_{ijk} + v\cdot\nabla\zeta_{ijk} - \Lambda\psi_{ijk} = f_1^{ijk}, \\ \partial_t\psi_{ijk} + v\cdot\nabla\psi_{ijk} + a\zeta_{ijk} = f_2^{ijk}, \end{cases}$$

where v is given in Lemma 6.1, $a = 1+\delta^2 Z_t + \delta^2 v\cdot\nabla Z$, and f_1^{ijk} and f_2^{ijk} are corrections of lower order terms. Applying the energy estimate to this system of quasilinear equations, we obtain the uniform boundedness of the solution stated in Theorem 2.1.

The details will be published elsewhere.

References

[1] G.B. Airy, Tides and waves, Encyclopaedia metropolitana, London, **5** (1845), 241–396.

[2] B. Alvarez-Samaniego and D. Lannes, Large time existence for 3D water-waves and asymptotics, Invent. Math., **171** (2008), 485–541.

[3] K.O. Friedrichs, On the derivation of the shallow water theory, Appendix to: "The formulation of breakers and bores" by J. J. Stoker in Comm. Pure Appl. Math., **1** (1948), 1–87.

[4] T. Iguchi, A shallow water approximation for water waves, submitted.

[5] T. Kano, Une théorie trois-dimensionnelle des ondes de surface de l'eau et le développement de Friedrichs, J. Math. Kyoto Univ., **26** (1986), 101–155 and 157–175 [French].

[6] T. Kano and T. Nishida, Sur les ondes de surface de l'eau avec une justification mathématique des équations des ondes en eau peu profonde, J. Math. Kyoto Univ., (1979) **19**, 335–370 [French].

[7] T. Kano and T. Nishida, Water waves and Friedrichs expansion. Recent topics in nonlinear PDE, 39–57, North-Holland Math. Stud., 98, North-Holland, Amsterdam, 1984.

[8] D. Lannes, Well-posedness of the water-waves equations, J. Amer. Math. Soc., **18** (2005), 605–654.

[9] Y.A. Li, A shallow-water approximation to the full water wave problem, Comm. Pure Appl. Math., **59** (2006), 1225–1285.

[10] V.I. Nalimov, The Cauchy-Poisson problem, Dinamika Splošn. Sredy, **18** (1974), 104–210 [Russian].

[11] L.V. Ovsjannikov, To the shallow water theory foundation, Arch. Mech., **26** (1974), 407–422.

[12] L.V. Ovsjannikov, Cauchy problem in a scale of Banach spaces and its application to the shallow water theory justification. Applications of methods of functional analysis to problems in mechanics, 426–437. Lecture Notes in Math., 503. Springer, Berlin, 1976.

[13] J.J. Stoker, Water waves: the mathematical theory with application, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York.

[14] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 2-D, Invent. Math., **130** (1997), 39–72.

[15] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, J. Amer. Math. Soc., **12** (1999), 445–495.

[16] H. Yosihara, Gravity waves on the free surface of an incompressible perfect fluid of finite depth, Publ. RIMS Kyoto Univ., **18** (1982), 49–96.